Boundary of maximal monotone operatos values*

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Abstract

We characterize the boundary of the values of maximal monotone operators defined in Hilbert spaces, by means only of the values at nearby points, which are closed enough to the reference point but distinct of it. This allows to write the values of such operators using finite convex (2-)combinations of the values at such nearby points. We also provide similar characterizations for the normal cone to prox-regular sets.

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1 Introduction

Given a continuous convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, according to [5, Theorem 3.1] the topological boundary of the Fenchel subdifferential of φ is completely characterized by means of the values of such subdifferential mapping at points, which are closed enough to the reference point but distinct of it. More specifically, for every $x \in \mathbb{R}^n$ we have that

$$\operatorname{bd}(\partial\varphi(x)) = \operatorname*{Limsup}_{y \longrightarrow_{\neq} x} \partial\varphi(y). \tag{1.1}$$

This characterization has been shown useful for many stability purposes of parametrized semi-infinite linear programming problems, given in $\mathbb{R}^n$ as ([8])

$$P(c,a,b) : \begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & a_t'x \le b_t,\ t \in T, \end{array}$$

for compact space T and continuous functions a and b on T. The characterization above was the main ingredient in [4, 5, 6] to derive point-based explicit expressions for the so-called *calmness moduli* of the associated feasible and optimal solutions set-valued mappings; we refer to [9, 10, 11]

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for more details on this calmness property. For instance, if $\mathcal{F}_a : C(T,\mathbb{R}) \to \mathbb{R}^n$ denotes the feasible set-valued mapping,

$$\mathcal{F}_a(b) := \{x \in \mathbb{R}^n : a_t' x \le b_t \ \forall t \in T\},$$

then the calmness modulus of $\mathcal{F}_a$ at a point $(\bar{b}, \bar{x})$ in its graph, given implicitly as

$$\operatorname{clm} \mathcal{F}_a(\bar{b}, \bar{x}) := \limsup_{\substack{x \to \bar{x},\ b \to \bar{b} \\ x \in \mathcal{F}_a(b)}} \frac{d(x, \mathcal{F}_a(\bar{b}))}{d(b, \bar{b})},$$

is rewritten in the more explicit form (using the convention $\frac{1}{0} = +\infty$)

$$\operatorname{clm} \mathcal{F}_a(\bar{b}, \bar{x}) = \left(\liminf_{x \to \bar{x},\ s(x) > 0} d_*(0, \partial s(x)) \right)^{-1},$$

where $s : \mathbb{R}^n \to \mathbb{R}$ is the convex continuous function given by

$$s(x) := \max_{t \in T} \{a_t' x - b_t\},$$

and whose subdifferential mapping can be easily estimated by means only of the data vectors a and b. From a qualitative point of view, the calmness of the mapping $\mathcal{F}_a$, say $\operatorname{clm} \mathcal{F}_a(\bar{b}, \bar{x}) > 0$, is equivalent to the fact that the function s has an (global) error bound at $\bar{x}$ (see [12, 13]).

At this stage, if, in addition, the set $\mathcal{F}_a(\bar{b})$ turns to be the singleton $\{\bar{x}\}$, in which case $s(x) > 0$ iff $x \ne \bar{x}$, then formula (1.1) goes into the play and entails a point-based expression of the calmness modulus of the mapping $\mathcal{F}_a$, that is given by

$$\operatorname{clm} \mathcal{F}_a(\bar{b}, \bar{x}) = (d_*(0, \operatorname{bd}(\partial s(\bar{x}))))^{-1}.$$

It is worth observing that in the framework of semi-infinite linear programming problems, this singleton's assumption is required for the solutions set-valued mapping and not for the feasible set-valued mapping (see [4, 5, 6] for more details).

For the aim of adapting this kind of analysis in a further research to more general semi-infinite linear programming problems with a non-necessarily compact index set T, so that the function s above lacks to be continuous, we extend in this paper formula (1.1) to the class of proper and lower semicontinuous convex functions. More generally, we establish similar characterizations for maximal monotone operators in the setting of Hilbert spaces. The first result given in Theorem 3.1 asserts that given a maximal monotone operator $A : H \rightrightarrows H$, for all $x \in H$ we have that

$$\operatorname{bd}(Ax) = \operatorname*{Limsup}_{y \longrightarrow_{\neq} x} \operatorname{bd}(Ay) = \operatorname*{Limsup}_{y \longrightarrow_{\neq} x} Ay,$$

where the Limsup is taken with respect to the norm. As a consequence, we prove that the value of A at x can be expressed using only different nearby points in the sense that for every $x \in H$ such that $\operatorname{bd}(Ax) \ne \emptyset$ it holds (Theorem 3.3)

$$Ax = \mathrm{N}_{\operatorname{cl}(\operatorname{dom} A)}(x) + \operatorname{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} Ay \right\},$$

where co_2 is the set of all the segments generated by the elements of the underlying set, and $\mathrm{N}_{\operatorname{cl}(\operatorname{dom} A)}(x)$ is the normal cone in the sense of convex analysis to the closure of the domain of the

operator A. Characterizations of similar type are given for the faces of the values of A, see Theorem 3.2. Extensions to nonconvex objects, as prox-regular sets and functions, is also considered in Theorems 4.1 and 4.2.

This paper is organized as follows: After Section 2, dedicated to present the necessary notations and the preliminary tools, we give the main result in Section 3: Theorem 3.1 characterizes the boundary of the values of maximal monotone operators, while Theorem 3.3 recovers the values of such operators using these boundary points. Theorem 3.2 specifies such characterizations to the faces of the values of maximal monotone operators. In Section 4 we extend this analysis to nonconvex objects, which are the normal cone to prox-regular sets (Theorem 4.1) and the subdifferential of prox-regular functions with uniform parameters (Theorem 4.2).

2 Notations and preliminary results

In this paper, H is a Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and associated norm $||\cdot||$. The weak topology on H is denoted by ω, while the strong and weak convergences in H are denoted by $\to$ and $\rightharpoonup$, resp. We denote by $\mathrm{B}(x, \rho)$ the closed ball with center $x \in H$ and radius $\rho > 0$; in particular, we write $\mathrm{B}_\rho := \mathrm{B}(\theta, \rho)$. The null vector in H is denoted θ. Given a set $S \subset H$, $\mathrm{co}\{S\}$ and $\mathrm{co}_2\{S\}$ are respectively the *convex hull* of S and the set

$$\mathrm{co}_2\, S := \{\alpha s_1 + (1-\alpha)s_2 : \alpha \in [0,1], s_1, s_2 \in S\}.$$

Observe that $\mathrm{co}_2\, S$ coincides with $\mathrm{co}\, S$ when $H = \mathbb{R}$, but the two sets may be different in general. By $\mathrm{int}(S)$, $\mathrm{bd}(S)$ and $\mathrm{cl}(S)$ (or, indistinctly, $\overline{S}$), we denote the *interior,* the *boundary* and the *closure* of S, respectively. The *indicator, the support* and the *distance functions* to the set S are respectively given by

$$\mathrm{I}_S(x) := 0 \text{ if } x \in S; +\infty \text{ if not}, \sigma_S(x) := \sup\{\langle x, s\rangle : s \in S\},\ d_S(x) := \inf\{||x-y|| :\ y \in S\}$$

(in the sequel we shall adopt the convention $\inf_\emptyset = +\infty$). We shall write $\xrightarrow{S}$ for the convergence when restricted to the set S, and $y \longrightarrow_{\neq} x$ when $y \to x$ with $y \neq x$. We denote Π_S the (*orthogonal) projection mapping* onto S defined as

$$\Pi_S(x) := \{y \in S :\ ||x-y|| = d_S(x)\}.$$

Next, we review some classical facts about convex functions and monotone operators; we refer to [3, 17] for more details. Given a function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$, we say that φ is proper if its domain $\mathrm{dom}\, \varphi := \{x \in H : \varphi(x) < +\infty\}$ is nonempty, lower semicontinuous (lsc, for short) if its epigraph $\mathrm{epi}\, \varphi := \{(x, \lambda) \in H \times \mathbb{R} : \varphi(x) \leq \lambda\}$ is closed, and convex if its epigraph is convex. If φ is convex, the Fenchel sudifferential mapping of φ as $x \in \mathrm{dom}\, \varphi$ is defined as

$$\partial\varphi(x) := \{x^* \in H : \langle x^*, y-x\rangle \leq \varphi(y) - \varphi(x) \ \ \forall y \in H\},$$

and $\partial\varphi(x) := \emptyset$ when $x \notin \mathrm{dom}\, \varphi$. The normal cone to a closed convex set $S \subset H$ is $\mathrm{N}_S(x) := \partial \mathrm{I}_S(x)$ for $x \in H$.

Given a set-valued operator $A : H \rightrightarrows H$, the *domain* and the *graph* of A are given by

$$\mathrm{dom}\, A := \{x \in H : Ax \neq \emptyset\},\ \mathrm{Gr}\, A := \{(x, x^*) : x^* \in Ax\}.$$

The operator A is said to be *monotone* if

$$\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0 \quad \text{for all} \;\; (x_1, x_1^*),\; (x_2, x_2^*) \in \operatorname{Gr} A,$$

and *maximal monotone* if, in addition, A coincides with every monotone operator containing its graph. In such a case, it is known that $\operatorname{cl}(\operatorname{dom} A)$ is convex, and that Ax is convex and closed for every $x \in H$. Hence, the *minimal norm* element of Ax; that is,

$$A^\circ x := \{x^* \in Ax : \|x^*\| = \min_{z^* \in Ax} \|z^*\|\},$$

is well-defined and unique.

Finally, given multifunction $F : H \rightrightarrows H$ we denote

$$\operatorname*{Limsup}_{y \longrightarrow x} F(y) := \{x^* \in H : \exists y_n \longrightarrow x,\; y_n^* \to x^*,\; \text{s.t. } y_n^* \in F(y_n)\ \forall n \geq 1\},$$

$$\operatorname*{Limsup}_{y \rightharpoonup x} F(y) := \{x^* \in H : \exists y_n \rightharpoonup x,\; y_n^* \to x^*,\; \text{s.t. } y_n^* \in F(y_n)\ \forall n \geq 1\},$$

$$\omega - \operatorname*{Limsup}_{y \longrightarrow x} F(y) := \{x^* \in H : \exists y_n \longrightarrow x,\; y_n^* \rightharpoonup x^*,\; \text{s.t. } y_n^* \in F(y_n)\ \forall n \geq 1\}.$$

3 Boundary of maximal monotone operators

In this section, we give the desired property which expresses a given maximal monotone operator $A : H \rightrightarrows H$, defined on a Hilbert space H, by means of its values at nearby points.

Definition 3.1 *Given $x \in \operatorname{dom} A$ and $v \in H$, we define the set $A(x; v) \subset H$ as*

$$A(x; v) := \{x^* \in Ax : \langle x^*, v \rangle = \sigma_{Ax}(v)\},$$

with the convention that $A(x, v) = \emptyset$ when $\sigma_{Ax}(v) = +\infty$.

Since Ax, $x \in \operatorname{dom} A$, is convex and closed, $A(x; \cdot)$ coincides with the subdifferential mapping of the proper, convex and lsc support function σ_{Ax}. As a consequence, the following remark resumes some easy properties of the set $A(x; v)$.

Remark 3.1 *Given $x \in \operatorname{dom} A$ and $v \in H$, we have:*

(i) $A(x; v)$ is convex and closed (possibly empty), and nonempty whenever the set Ax is bounded.

(ii) $A(x; \theta) = Ax$, and if $v \neq \theta$ then $A(x; v)$ is a subset of $\operatorname{bd}(Ax)$. In the last case, we refer to $A(x; v)$ as the face of Ax with respect to the direction v.

(iii) $A(x; \alpha v) = A(x; v)$ for any $v \neq \theta$ and $\alpha > 0$; thus, the face $A(x; v)$ depends only on the direction v.

We shall need the following lemma.

Lemma 3.1 (see, e.g., [7]) *For any nonempty closed convex set $S \subset H$, the set of points $s \in \operatorname{bd}(S)$ such that $\mathrm{N}_S(s) \neq \{\theta\}$ is dense in $\operatorname{bd}(S)$.*

Proposition 3.1 *Let $x \in \operatorname{dom} A$ and $v \neq \theta$ be given. Then we have that*

$$\operatorname{bd}(Ax) = \operatorname{cl}\left(\bigcup_{v\neq\theta} A(x;v)\right).$$

Proof. The inclusion " $\supset$ " being obvious, due to the definition of the set $A(x;v)$, we only need to prove the inclusion " $\subset$ ". Take an arbitrary vector $\xi \in \operatorname{bd}(Ax)$. According to Lemma 3.1, there exists a sequence $(\xi_n)_n \subset \operatorname{bd}(Ax)$ such that $\xi_n \to \xi$ and $\mathrm{N}_{Ax}(\xi_n) \neq \{\theta\}$. Hence, for each n there exists $v_n \neq \theta$ such that $v_n \in \mathrm{N}_{Ax}(\xi_n) = \partial \mathrm{I}_{Ax}(\xi_n)$, or, equivalently, $\xi_n \in \partial\sigma_{Ax}(v_n) = A(x;v_n)$; that is, $\xi \in \operatorname{cl}\left(\bigcup_{v\neq\theta} A(x;v)\right)$. ■

Theorem 3.1 *For every $x \in H$ we have*

$$\operatorname{bd}(Ax) = \operatorname*{Limsup}_{y \to_{\neq} x} \operatorname{bd}(Ay) = \operatorname*{Limsup}_{y \to_{\neq} x} Ay.$$

Proof. To prove the first statement of the theorem we proceed by verifying the following inclusions, for every fixed $x \in H$,

$$\operatorname{bd}(Ax) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \operatorname{bd}(Ay) \subset \operatorname*{Limsup}_{y \to_{\neq} x} Ay \subset \operatorname{bd}(Ax). \tag{3.1}$$

First, we observe that when $x \notin \operatorname{dom} A$, these inclusions follows since that, using the norm-weak upper semicontinuity of the (maximal monotone) operator A,

$$\emptyset = \operatorname{bd}(Ax) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \operatorname{bd}(Ay) \subset \operatorname*{Limsup}_{y \to_{\neq} x} Ay \subset Ax = \emptyset.$$

So, we may assume that $x \in \operatorname{dom} A$. Also, if $\operatorname{bd}(Ax) = \emptyset$, then we would have that $Ax = H$, so that $\operatorname{dom} A = \{x\}$ and this leads to

$$\operatorname*{Limsup}_{y \to_{\neq} x} \operatorname{bd}(Ay) = \operatorname*{Limsup}_{y \to_{\neq} x} Ay = \emptyset;$$

that is, the conclusion of the first statement is also true in this case.

From the observation above we assume now that $\operatorname{bd}(Ax) \neq \emptyset$. Take $x^* \in \operatorname{bd}(Ax)$ $(\subset Ax)$. According to Lemma 3.1, for each $n \geq 1$ there exists $x_n^* \in \operatorname{bd}(Ax)$ such that $\|x_n^* - x^*\| \leq \frac{1}{n}$ and $\mathrm{N}_{Ax}(x_n^*) \neq \theta$; hence, $x_n^* = \Pi_{Ax}(v_n)$ for some $v_n \in H \setminus Ax$. We fix $n \geq 1$ and consider the following differential inclusion

$$\dot{z}(t) \in v_n - Az(t) \ \ t \in [0,1], \ \ z(0) = x,$$

which (see, e.g., [3]) possesses a unique solution $z_n(\cdot)$ that satisfies $z_n(t) \in \operatorname{dom} A$ for all $t \in [0,1]$, and such that the function

$$t \mapsto \frac{d^+ z_n(t)}{dt} = (v_n - Az_n(t))^\circ = v_n - \Pi_{Az_n(t)}(v_n) \tag{3.2}$$

is right-continuous on $[0,1)$. In particular, one has

$$\frac{d^+ z_n(0)}{dt} = (v_n - Az_n(0))^\circ = (v_n - Ax)^\circ = v_n - \Pi_{Ax}(v_n) = v_n - x_n^*;$$

hence, since $v_n - x_n^* \neq \theta$, we get $z_n(t) \neq x$ for all small $t \in [0,1)$. Then, from the right-continuity of $\frac{d^+ z_n(\cdot)}{dt}$ and the expressions in (3.2), there exists a sequence $t_k \downarrow 0$ such that $z_{n,k}^* := \Pi_{Az_n(t_k)}(v_n) \to x_n^*$ as k goes to $+\infty$, and $z_n(t_k) \neq x$ for all $k \geq 1$. We observe that $z_{n,k}^* \in \mathrm{bd}(Az_n(t_k))$ for all $k \geq 1$ is a cofinite set, because for otherwise, since $z_{n,k}^* \in Az_n(t_k)$ we would have $z_{n,k}^* \in \mathrm{int}(Az_n(t_k))$ for all k in a cofinite set K, and this would lead to $v_n \in Az_n(t_k)$ for all $k \in K$. Consequently, as $z_n(t_k) \to x$ when k goes to $+\infty$, the maximal monotonicity of A would give us $v_n \in Ax$, which is a contradiction. Now, we may choose a diagonal sequence $(z_{n,k_n}^*)_n$ such that $z_{n,k_n}^* \to x^*$ as $n \to +\infty$, and this shows that $x^* \in \mathrm{Limsup}_{y \to_{\neq} x} \mathrm{bd}(Ay)$, which yields the first inclusion in (3.1).

We take now $x^* \in \mathrm{Limsup}_{y \to_{\neq} x} Ay$, so that $x^* = \lim\limits_{n\to\infty} x_n^*$ for some $x_n^* \in Ax_n$ with $x_n \to x$ and $x_n \neq x$. Then by the norm-weak upper semicontinuity of the operator A, we deduce that $x^* \in Ax$. Thus, it suffices to prove that $x^* \in H \setminus \mathrm{int}(Ax)$. Proceeding by contradiction, we assume that $x^* + r\mathbb{B} \subset Ax$ for some $r > 0$. Then, using the monotonicity of A, for every $n \geq 1$ one has that

$$\langle x_n^* - \left(x^* + r\frac{x_n - x}{\|x_n - x\|}\right), x_n - x\rangle \geq 0,$$

which gives

$$\|x_n^* - x^*\|\|x_n - x\| \geq \langle x_n^* - x^*, x_n - x\rangle \geq \langle r\frac{x_n - x}{\|x_n - x\|}, x_n - x\rangle = r\|x_n - x\|;$$

that is, $\|x_n^* - x^*\| \geq r$ for every $n \geq 1$, and this contradicts the convergence of (x_n^*) to x^*. Hence, $x^* \in \mathrm{bd}(Ax)$ and we conclude the proof of (3.1). ■

It easily follows from Theorem 3.1 that

$$\mathrm{bd}(Ax) \subset \operatorname*{Limsup}_{y \to_{\neq} x} Ay \subset \omega - \operatorname*{Limsup}_{y \to_{\neq} x} Ay,$$

but the last inclusion may be strict, as the following example shows.

Example 3.1 Assume that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis for H, and consider the maximal monotone operator $A := \partial\|\cdot\|$. So,

$$A\theta = \mathrm{B}(\theta, 1) \text{ and } Ax = \frac{x}{\|x\|} \text{ for all } x \neq \theta.$$

We observe that the sequence $(\frac{e_n}{n})_{n\in\mathbb{N}}$ strongly converges to θ, and

$$A\frac{e_n}{n} = e_n \rightharpoonup \theta \in \mathrm{int}(\mathrm{B}(\theta, 1)) = \mathrm{int}(A\theta).$$

We give an interesting corollary of Theorem 3.1.

Corollary 3.1 *For every $x \in H$ we have*

$$d(\theta, \mathrm{bd}(Ax)) = \liminf_{y \to_{\neq} x} d(\theta, Ay).$$

Consequently, if x is such that $\theta \notin \mathrm{int}(Ax)$, then

$$\|A^\circ x\| = \liminf_{y \longrightarrow_{\neq} x} \|A^\circ y\|.$$

Proof. It suffices to consider the case when $x \in \operatorname{dom} A$, because otherwise both sides of the equality are equal to $+\infty$.

We may distinguish two cases: If $\theta \notin Ax$, then $d(\theta, \operatorname{bd}(Ax)) = d(\theta, Ax) = \|A^\circ x\|$. Thus, according to Theorem 3.1 there are sequences (y_n), $(y_n^*) \subset H$ such that

$$y_n \rightarrow_{\neq} x,\; y_n^* \in Ay_n,\; \text{and}\; y_n^* \rightarrow A^\circ x \text{ as } n \rightarrow +\infty.$$

Hence,

$$\|A^\circ x\| = \lim_{n\to\infty} \|y_n^*\| \geq \liminf_{n\to\infty} d(\theta, Ay_n) \geq \liminf_{y \rightarrow_{\neq} x} d(\theta, Ay),$$

and so $d(\theta, \operatorname{bd}(Ax)) = \|A^\circ x\| \geq \liminf_{y\rightarrow_{\neq} x} d(\theta, Ay)$. Hence, if $\liminf_{y\rightarrow_{\neq} x} d(\theta, Ay) = +\infty$, then the first equality of the corollary obviously. Otherwise, we suppose that $\liminf_{y\rightarrow_{\neq} x} d(\theta, Ay) < \alpha$ for some $\alpha \in \mathbb{R}$, and let sequences (y_n), $(y_n^*) \subset H$ be such that

$$y_n \rightarrow_{\neq} x,\; y_n^* \in Ay_n,\; \text{and}\; \lim_{n\to\infty} \|y_n^*\| < \alpha.$$

Thus, taking into account Theorem 3.1, we may suppose that $y_n^* \rightarrow x^* \in \operatorname{bd}(Ax)$; that is,

$$d(\theta, \operatorname{bd}(Ax)) \leq \|x^*\| \leq \alpha.$$

We get the desired inequality "$\leq$" when α goes to $\liminf_{y\rightarrow_{\neq} x} d(\theta, Ay)$, and this completes the proof of the first statement.

To prove the last statement, we observe that under the current assumption, we have that $\|A^\circ x\| = d(\theta, Ax) = d(\theta, \operatorname{bd}(Ax))$, and so it suffices to use the first statement of the theorem. ■

Corollary 3.2 *For every $x \in H$ such that Ax is a nonempty bounded set, we have*

$$\|Ax\| \leq \limsup_{y \longrightarrow_{\neq} x} \|Ay\|,$$

and, when H is finite-dimensional,

$$\|Ax\| = \limsup_{y \longrightarrow_{\neq} x} \|Ay\|.$$

Proof. Let $x \in H$ be as in the corollary. Then for any $\varepsilon > 0$ there exists $x^* \in \operatorname{bd}(Ax)$ such that $\|x^*\| \geq \|Ax\| - \varepsilon$. According to Theorem 3.1, there exist sequences $y_n \rightarrow x$ and $y_n^* \in Ay_n$ such that $y_n \neq x$ and $y_n^* \rightarrow x^*$ as $n \rightarrow +\infty$. Thus,

$$\limsup_{y \longrightarrow_{\neq} x} \|Ay\| \geq \limsup_{n\to+\infty} \|Ay_n\| \geq \lim_{n\to\infty} \|y_n^*\| = \|x^*\| \geq \|Ax\| - \varepsilon,$$

and the desired inequality follows when ε goes to 0.

We assume now that H is finite-dimensional, so that according to the first statement we only need to prove that

$$\|Ax\| \geq \limsup_{y \longrightarrow_{\neq} x} \|Ay\|.$$

Indeed, if $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| = +\infty$, then since A is locally bounded in $\operatorname{int}(\operatorname{cl}(\operatorname{dom} A))$ (when this set is nonempty), it follows that $x \in \operatorname{bd}(\operatorname{cl}(\operatorname{dom} A))$. Hence, $\mathrm{N}_{\operatorname{cl}(\operatorname{dom} A)}(x) \neq \{\theta\}$ and the equality $Ax = Ax + \mathrm{N}_{\operatorname{cl}(\operatorname{dom} A)}(x)$, which comes from the maximality of the operator A, entail the contradiction $\|Ax\| = +\infty$. Consequently, we may suppose that $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| < +\infty$. We let a sequence $(y_n, y_n^*)_n \subset \operatorname{Gr} A$ be such that $y_n \to x$, $y_n \neq x$ and $\limsup_{y \longrightarrow_{\neq} x} \|Ay\| = \lim_{n \to \infty} \|y_n^*\|$. We may also assume that the sequence $(y_n^*)_n$ converges to some $x^* \in Ax$. Then

$$\|Ax\| \geq \|x^*\| = \lim_{n \to \infty} \|y_n^*\| = \limsup_{y \longrightarrow_{\neq} x} \|Ay\|,$$

as we wanted to prove. ■

The following result concerns the faces of the values of maximal monotone operators.

Theorem 3.2 *For every $x \in \operatorname{dom} A$ and $v \neq \theta$ we have*

$$A(x; v) = \operatorname*{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw) = \operatorname*{Limsup}_{w \rightharpoonup v,\ t \downarrow 0} A(x + tw) = \omega\text{-}\operatorname*{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw).$$

Proof. We fix $x \in \operatorname{dom} A$ and $v \neq \theta$, and take $x^* \in A(x; v)$. From Definition 3.1, we have that $v \in (\partial \sigma_{Ax})^{-1}(x^*) = \mathrm{N}_{Ax}(x^*)$, which ensures that $x^* = \Pi_{Ax}(x^* + v)$. Let us consider the following differential inclusion

$$\dot{z}(t) \in x^* + v - Az(t) \ \ t \geq 0,\ \ z(0) = x.$$

As in the proof of Theorem 3.1, this differential inclusion has a unique solution $z(\cdot)$ such that

$$\lim_{t \downarrow 0} \frac{d^+ z(t)}{dt} = \lim_{t \downarrow 0} (x^* + v - Az(t))^\circ = \frac{d^+ z(0)}{dt} = (x^* + v - Ax)^\circ = (x^* + v) - x^* = v. \tag{3.3}$$

We denote

$$x_n^* := \Pi_{Az(\frac{1}{n})}(x^* + v),\ \ w_n := \frac{z(\frac{1}{n}) - x}{\frac{1}{n}};$$

hence, (3.3) ensures that $\frac{d^+ z(\frac{1}{n})}{dt} = (x^* + v - Az(\frac{1}{n}))^\circ = x^* + v - x_n^* \to \frac{d^+ z(0)}{dt} = v$. Therefore, as $n \to +\infty$ we obtain that

$$x_n^* \to x^*,\ \ w_n \to \frac{d^+ z(0)}{dt} = v,$$

and so

$$x^* = \lim_{n \to \infty} x_n^* \subset \operatorname*{Limsup}_{n \to \infty} Az(\frac{1}{n}) = \operatorname*{Limsup}_{n \to \infty} A(x + \frac{1}{n} w_n) \subset \operatorname*{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw),$$

showing that

$$A(x; v) \subset \operatorname*{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw) \subset \operatorname*{Limsup}_{w \rightharpoonup v,\ t \downarrow 0} A(x + tw).$$

Thus, since $A(x; v) \subset \operatorname{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw) \subset \omega - \operatorname{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw)$, we only need to verify that

$$\operatorname*{Limsup}_{w \rightharpoonup v,\ t \downarrow 0} A(x + tw) \subset A(x; v) \text{ and } \omega - \operatorname*{Limsup}_{w \to v,\ t \downarrow 0} A(x + tw) \subset A(x; v). \tag{3.4}$$

To see the first inclusion, we take $x^* \in \operatorname{Limsup}_{w \rightharpoonup v,\, t\downarrow 0} A(x+tw)$, so that $x^* = \lim_n x_n^*$ for some sequences (x_n^*), $(w_n) \in H$, $(t_n) \subset \mathbb{R}_+$, such that $x_n^* \in A(x+t_n w_n)$, $w_n \rightharpoonup v$, and $t_n \downarrow 0$. It follows by the maximal monotonicity of A that $x^* \in Ax$, and for all $\xi \in Ax$

$$\langle x_n^* - \xi, w_n \rangle = \frac{1}{t_n} \langle x_n^* - \xi, x + t_n w_n - x \rangle \geq 0.$$

So, by taking the limit as $n \to +\infty$ we obtain that $\langle x^*, v\rangle \geq \sup_{\xi \in Ax} \langle \xi, v \rangle \geq \langle x^*, v \rangle$, which shows that $x^* \in A(x; v)$, and the first inclusion in (3.4) follows. We conclude the proof of the theorem because the second inclusion in (3.4) can be obtained using the same arguments as in the first inclusion. ■

The following example shows the necessity of moving the vector v in the expression of Theorem 3.2.

Example 3.2 Consider the maximal monotone operator A defined on H as

$$Ax := x + \mathrm{N}_{\mathrm{B}(\theta,1)}(x),$$

and let $x,\, v \in H \setminus \{\theta\}$ be such that

$$\|x\| = 1 \text{ and } \langle v, x \rangle = 0.$$

Then one can easily check that $Ax = [1, +\infty[\ x$, and so

$$A(x; v) = \left\{ x^* \in Ax : \langle x^*, v \rangle = \sup_{\xi \in Ax} \langle \xi, v \rangle = \sup_{\alpha \in [1,+\infty[} \langle \alpha x, v \rangle = 0 \right\} = Ax.$$

But for any $t > 0$ we have that $A(x+tv) = \emptyset$, which shows that

$$\omega - \operatorname*{Limsup}_{t\downarrow 0} A(x+tv) = \operatorname*{Limsup}_{t\downarrow 0} A(x+tv) = \emptyset.$$

In Theorem 3.3 we give the expression of the values of maximal monotone operators by using the values at nearby points. We need first to check the following lemma.

Lemma 3.2 *Given $x \in \operatorname{dom} A$, for every $x^* \in Ax$ it holds*

$$\mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) = \{v \in H : x^* + tv \in Ax,\ \forall t \geq 0\} =: d_\infty(Ax). \tag{3.5}$$

Proof. Since the operator $A + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}$ is monotone and $\operatorname{Gr} A \subset \operatorname{Gr}(A + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)})$, the maximality of A ensures that $Ax + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) = Ax$, which implies that $\mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) \subset d_\infty(Ax)$. Take now $v \in d_\infty(Ax)$, so that $x^* + tv \in Ax$ for all $t \geq 0$. Then, by the monotonicity of A we get

$$\langle y^* - (x^* + tv), y - x \rangle \geq 0\ \forall y^* \in Ay,\ \forall t \geq 0,$$

which in turn leads to

$$\langle y^* - x^*, y - x \rangle \geq t \langle v, y - x \rangle\ \forall y^* \in Ay,\ \forall t \geq 0.$$

Hence, $\langle v, y - x \rangle \leq 0$ for every $y \in \operatorname{dom} A$, and we deduce that $v \in \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x)$. ■

Theorem 3.3 *For every $x \in \operatorname{dom} A$ such that $\operatorname{bd}(Ax) \neq \emptyset$ we have that*

$$Ax = \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} Ay \right\}.$$

Proof. First, according to Theorem 3.1, ensuring that $\operatorname{bd}(Ax) = \operatorname{Limsup}_{y \to_{\neq} x} Ay$, and to the maximal monotonicity of the operator A, ensuring that $A = A + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}$, we only need to prove the following inclusion when $\operatorname{int}(Ax) \neq \emptyset$,

$$\operatorname{int}(Ax) \subset \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2 \{\operatorname{bd}(Ax)\}. \tag{3.6}$$

Given $x^* \in \operatorname{int}(Ax)$, we fix $x_0^* \in \operatorname{bd}(Ax)$ and introduce the set

$$S := \{x_0^* + t(x^* - x_0^*) : t \geq 1\}.$$

On the one hand, if $S \cap \operatorname{bd}(Ax) = \emptyset$, then $S \subset Ax$ and, due to the convexity of Ax, we obtain $x_0^* + \mathbb{R}_+(x^* - x_0^*) \subset Ax$. Hence, thanks to Lemma 3.2 we deduce that $x^* - x_0^* \in \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x)$, and so we get

$$x^* \in x_0^* + \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) \subset \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2\{\operatorname{bd}(Ax)\},$$

which yields (3.6). On the other hand, if $S \cap \operatorname{bd}(Ax) \neq \emptyset$, then there exists some $t > 1$ such that $z^* = x_0^* + t(x^* - x_0^*) \in \operatorname{bd}(Ax)$. Thus, we get

$$x^* = \frac{1}{t} z^* + (1 - \frac{1}{t}) x_0^* \in \mathrm{co}_2\{\operatorname{bd}(Ax)\} \subset \mathrm{N}_{\mathrm{cl}(\operatorname{dom} A)}(x) + \mathrm{co}_2\{\operatorname{bd}(Ax)\},$$

and this completes the proof of the theorem. ∎

4 Prox-regular analysis

In this section, we extend the results of the previous section to two classes of operators of nonsmooth analysis, the normal cone to uniformly r-prox-regular sets, and the class of prox-regular extended-real-valued functions with uniform parameters. As before, we work in the setting of a given Hilbert space H.

We start by giving the definition of the proximal normal cone.

Definition 4.1

([7]) *Given a set $C \subset H$ and $x \in C$, the proximal normal cone to C at x, denoted by $\mathrm{N}_C^P(x)$, is the set of vectors $x^* \in H$ for which there exists $m > 0$ such that*

$$\langle x^*, y - x \rangle \leq m \left\| y - x \right\|^2 \text{ for all } y \in C.$$

Definition 4.2 ([14]) *For positive numbers r and α, a closed set C is said to be (r, α)-prox-regular at $\overline{x} \in C$ provided that one has $x = \Pi_C(x + v)$, for all $x \in C \cap \mathrm{B}(\overline{x}, \alpha)$ and all $v \in \mathrm{N}_C^P(x)$ such that $||v|| < r$. The set C is r-prox-regular (resp., prox-regular) at $\overline{x}$ when it is (r, α)-prox-regular at $\overline{x}$ for some real $\alpha > 0$ (resp., for some numbers $r, \alpha > 0$). The set C is said to be r-uniformly prox-regular when $\alpha = +\infty$.*

The following theorem describes the boundary set of the normal cone of a uniformly r-prox-regular set, by means of its values at nearby points, which are different from the reference point. We also characterize such normal cones by means of their boundaries points. Recall that the Bouligand tangent cone of a prox-regular closed set C at $x \in C$ is given by

$$\mathrm{T}_C(x) := \left(\mathrm{N}_C^P(x)\right)^*.$$

Theorem 4.1 *Let $C \subset H$ be a uniformly r-prox-regular set. Then for every $x \in C$ we have that*

$$\mathrm{bd}(\mathrm{N}_C^P(x)) = \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(\mathrm{N}_C^P(y)) = \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y). \tag{4.1}$$

If $\mathrm{int}(\mathrm{T}_C(x)) \neq \emptyset$*, then*

$$\mathrm{N}_C^P(x) = \mathrm{co}_2 \left\{\mathrm{bd}(\mathrm{N}_C^P(x))\right\} = \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y) \right\}. \tag{4.2}$$

Proof. First, we observe that the inclusions

$$\mathrm{bd}(\mathrm{N}_C^P(x)) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{bd}(\mathrm{N}_C^P(y)) \subset \operatorname*{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y), \tag{4.3}$$

follow as in the the proof of Theorem 3.1, since the following differential inclusion,

$$\dot{z}(t) \in f(z(t)) - \mathrm{N}_C^P(z(t)) \;\; t \in [0,1], \;\; z(0) = x \in C,$$

for a given Lipschitz function $f : H \to H$, also possesses a unique solution $z(\cdot)$ such that the function $\frac{d^+ z(\cdot)}{dt}$ is right-continuous on $[0,1[$ and $\frac{d^+ z(t)}{dt} = \left(f(z(t)) - \mathrm{N}_C^P(z(t))\right)^{\circ}$ for all $t \in [0,1[$ (see [1, Theorem 4.6] for more details).

We are going to prove the converse inclusions of (4.3). We take $\xi \in \mathrm{Limsup}_{y \to_{\neq} x} \mathrm{N}_C^P(y)$, and let the sequences (y_n) and (ξ_n) be such that

$$\xi_n \in \mathrm{N}_C^P(y_n), \;\; y_n \to x, \;\; \xi_n \to \xi \text{ as } n \to +\infty;$$

hence, we may suppose that for some $M > 0$ we have that $\xi_n \in \mathrm{N}_C^P(y_n) \cap \mathrm{B}_M$ for all $n \in \mathbb{N}$. Next, using the r-uniform prox-regularity of the set C, we obtain that $\xi \in \mathrm{N}_C^P(x)$ ([14]). We claim that $\xi \in \mathrm{bd}(\mathrm{N}_C^P(x))$. Proceeding by contradiction, we assume that for some positive number ρ such that $\rho < M$ it holds $\xi + \mathrm{B}_\rho \subset \mathrm{N}_C^P(x)$; that is,

$$\xi + \rho \frac{y_n - x}{\|y_n - x\|} \in \mathrm{N}_C^P(x) \; \forall n \in \mathbb{N}.$$

Now, using the monotonicity of the mapping $x \to \mathrm{N}_C^P(x) \cap \mathrm{B}_{2M} + \frac{2M}{r} x$ (see [14]), we get

$$\langle \xi_n + \frac{2M}{r} y_n - (\xi + \rho \frac{y_n - x}{\|y_n - x\|} + \frac{2M}{r} x), y_n - x \rangle \geq 0 \;\; \text{for all } n \geq 1,$$

which implies that

$$\|\xi_n - \xi\| \|y_n - x\| + \frac{2M}{r} \|y_n - x\|^2 \geq \langle \xi_n - \xi, y_n - x \rangle + \frac{2M}{r} \|y_n - x\|^2 \geq \rho \|y_n - x\|,$$

and, dividing by $\|y_n - x\|$,
$$\|\xi_n - \xi\| + \frac{2M}{r}\|y_n - x\| \geq \rho,$$
which is a contradiction. Hence, $\xi \in \mathrm{bd}(\mathrm{N}_C^P(x))$ and (4.3) holds as equalities.

In this last part of the proof, we assume that $\mathrm{int}(\mathrm{T}_C(x)) \neq \emptyset$; that is, there exist $v \in H$ and $\eta > 0$ such that $v + \mathrm{B}_\eta \subset \mathrm{int}(\mathrm{T}_C(x))$. According to the first statement of the theorem we only need to prove that
$$\mathrm{int}(\mathrm{N}_C^P(x)) \subset \mathrm{co}_2\left\{\mathrm{bd}(\mathrm{N}_C^P(x))\right\}. \tag{4.4}$$
We take $\xi \in \mathrm{int}(\mathrm{N}_C^P(x)) \setminus \{\theta\}$, so that $-\xi \notin \mathrm{N}_C^P(x)$ by [16, Exercise 9.42] (the proof of [16, Exercise 9.42] can be easily extended to the current infinite-dimensional setting), and hence we can choose $z^* \in \mathrm{bd}(\mathrm{N}_C^P(x)) \setminus \{\theta\}$. Let us show that for some $t_0 > 0$ we have that $\xi + t_0(\xi - t_0 z^*) \notin \mathrm{N}_C^P(x)$. Otherwise, $\xi + t(\xi - tz^*) \in \mathrm{N}_C^P(x)$ for all $t \geq 0$, and we get
$$\frac{1+t}{t^2}\xi - z^* \in \mathrm{N}_C^P(x) \;\; \forall t > 0,$$
which as $t \to +\infty$ gives us $-z^* \in \mathrm{N}_C^P(x)$, which contradicts the nonemptyness of the set $\mathrm{int}(\mathrm{T}_C(x))$ (again by [16, Exercise 9.42]). Then, there exists some $\beta \in (0,1)$ such that $w^* := \xi + \beta t_0(\xi - t_0 z^*) \in \mathrm{bd}(\mathrm{N}_C^P(x))$, and hence $\xi = \frac{1}{1+\beta t_0} w^* + \frac{\beta t_0}{1+\beta t_0}(t_0 z^*) \in \mathrm{co}_2\left\{\mathrm{bd}(\mathrm{N}_C^P(x))\right\}$. ■

In this last part of the paper, we extend the results of Section 3 to the proximal subdifferential mapping of lsc functions.

Definition 4.3 [2, Definition 3.1.] *Given a lsc function $f : H \to \mathbb{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$, a vector $x^* \in H$ is called proximal subgradient of f at x, written $x^* \in \partial_P f(x)$, if there are $\rho, \delta > 0$ such that*
$$f(y) \geq f(x) + \langle x^*, y - x\rangle - \delta||y - x||^2, \;\; \forall y \in \mathrm{B}(x, \rho).$$
A vector $x^* \in H$ is called limiting subgradient of f at x, written $\xi \in \partial_L f(x)$, if there are sequence (x_k), $(x_k^*) \subset H$ such that
$$x^* = \omega - \lim_{k\to\infty} x_k^*, \;\; x_k \longrightarrow x, \;\; f(x_k) \longrightarrow f(x), \;\; x_k^* \in \partial_P f(x_k).$$

Definition 4.4 [2, Definition 3.1.] *A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be prox-regular at $\bar{x} \in \mathrm{dom}\, f$ with uniform parameters if there exist $\varepsilon, r > 0$ such that for any $\bar{v} \in \partial_L f(\bar{x})$, one has, for all $(x, v) \in \mathrm{Gr}(\partial_L f)$ satisfying $\|x - \bar{x}\| < \varepsilon$, $|f(x) - f(\bar{x})| < \varepsilon$ and $\|v - \bar{v}\| < \varepsilon$,*
$$f(x') \geq f(x) + \langle v, x' - x\rangle - \frac{r}{2}\|x' - x\|^2 \;\; \forall x' \in \mathrm{B}(\bar{x}, \varepsilon).$$

It is worth observing that for prox-regular functions with uniform parameters f at $x \in \mathrm{dom}\, f$, we have that $\partial_P f(\bar{x}) = \partial_L f(\bar{x})$, and, in particular, if f is convex, then $\partial_P f(\bar{x}) = \partial f(\bar{x})$. In the following result, we give the counterpart of Theorem 3.1 to the proximal subdifferential mapping of prox-regular functions.

Theorem 4.2 *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lsc function and let $x \in \mathrm{dom}\, f$. If f is prox-regular with uniform parameters on a neighborhood of x with the same parameter $r > 0$, then*
$$\mathrm{bd}(\partial_P f(x)) = \operatorname*{Limsup}_{y \to_{\neq} x} \partial_P f(y),$$

and, provided that $\mathrm{bd}(\partial_P f(x)) \neq \emptyset$*,*

$$\partial_P f(x) = \mathrm{N}_{\mathrm{dom}\, f}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} \partial_P f(y) \right\}.$$

Proof. According to [2, Proposition 3.6], the current prox-regularity assumption entails the existence of an open convex neighborhood U of x and a lsc convex function g such that

$$f(y) = g(y) - \frac{r}{2}\|y\|^2 \ \ \forall y \in U; \tag{4.5}$$

hence, $\partial_P f(y) = \partial g(y) - ry$ for all $y \in U$. Thus, since ∂g is a maximal monotone operator [15], by applying Theorem 3.1 we get

$$\begin{aligned} \mathrm{bd}(\partial_P f(x)) &= \mathrm{bd}(\partial g(x) - rx) \\ &= \mathrm{bd}(\partial g(x)) - rx \\ &= \operatorname*{Limsup}_{y \to_{\neq} x} \partial g(y) - rx \\ &= \operatorname*{Limsup}_{y \to_{\neq} x} (\partial g(y) - ry) \\ &= \operatorname*{Limsup}_{y \to_{\neq} x} (\partial_P f(y)), \end{aligned}$$

which yields the first conclusion.

To prove the second statement we observe that $\mathrm{dom}\, f \cap U = \mathrm{dom}\, g \cap U$, which yields $\mathrm{N}_{\mathrm{dom}\, f}(x) = \mathrm{N}_{\mathrm{dom}\, g}(x)$. Thus, sine $\mathrm{bd}(\partial g(x)) = \mathrm{bd}(\partial_P f(x)) + rx \neq \emptyset$ due to the current assumption, by applying Theorem 3.3 and taking into account (4.5) we get

$$\begin{aligned} \partial_P f(x) &= \partial g(x) - rx \\ &= \mathrm{N}_{\mathrm{cl}(\mathrm{dom}\, \partial g)}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} (\partial g(y) - ry) \right\} \\ &= \mathrm{N}_{\mathrm{dom}\, f}(x) + \mathrm{co}_2 \left\{ \operatorname*{Limsup}_{y \to_{\neq} x} (\partial_P f(y)) \right\}, \end{aligned}$$

where we used the fact that $\mathrm{cl}(\mathrm{dom}\, \partial g) = \mathrm{cl}(\mathrm{dom}\, g)$ (see, e.g. [17]). ■

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